Chapter 5

Topological Entanglement Entropy

A topologically ordered system shows degenerate ground states below a gap, and this degeneracy, which depends on the topology of the entire system, cannot be ascribed to any type of conventional spontaneous symmetry breaking. Indeed, these degenerate ground states cannot be distinguished by any local observable, as explicitly demonstrated in a solvable model in Ref. [24] and in QDMs in Chap. 4. Preskill [62] suggested that this bizarre degeneracy can be regarded as a global encoding of information reminiscent of quantum error-correcting codes and is a consequence of some long-distance entanglement. A characterization of this global entanglement was realized recently by Kitaev and Preskill (KP) [31] and by Levin and Wen (LW) [41]. They argued that in a topological phase there is a universal additive constant in the entanglement entropy, called the topological entanglement entropy. This constant reflects the underlying gauge theory for the topological phase. By measuring this constant, one can detect the presence of topological order and moreover classify different topological phases. KP and LW illustrated this proposal using effective field theories and exactly solvable models.

In this chapter, we analyze the entanglement entropy in the QDM on the triangular lattice (presented in Sec. 2.5.3) and examine the effectiveness of the proposal in numerical calculations of finite-size systems. The dimer liquid phase in this model is known to have $\mathbb{Z}_2$ topological order [54]. We mainly consider the Rokhsar-Kivelson (RK) point [66], where the ground states are exactly known and where the calculation of reduced density matrices can be efficiently done as described in Appendix B. We examine the two original constructions to measure the topological entropy, and we observe that in the large-area limit they both approach the value expected for $\mathbb{Z}_2$ topological order. We also consider the entanglement entropy on a topologically non-trivial “zigzag” area and propose a way to measure the topological entropy accurately even in relatively small systems.

5.1 Entanglement entropy in quantum many-body systems

Exotic phenomena in quantum many-body systems are accompanied by non-trivial patterns of entanglement in ground-state wave functions. One useful measure of entanglement for a many-body state $|\Psi\rangle$ is the entanglement entropy $S_\Omega$ between a part $\Omega$ of the system and the rest of the system $\bar{\Omega}$. It is defined as the von Neumann entropy of the reduced density matrix $\rho_\Omega$:

$$S_\Omega = - \text{Tr} \rho_\Omega \ln \rho_\Omega, \quad \rho_\Omega = \text{Tr}_{\bar{\Omega}} |\Psi\rangle\langle\Psi|.$$  

The entanglement entropy $S_\Omega$ is zero for product states and is positive for others. Intuitively speaking, the entanglement entropy measures how “quantum” a given state is. The entanglement entropy has the following properties [57].

(i) The values of entanglement entropies are the same for an area $\Omega$ and its complement $\bar{\Omega}$:

$$S_\Omega = S_{\bar{\Omega}}.$$  

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(ii) Subadditivity: when an area $A$ is divided into $A_1$ and $A_2$, we have

$$S_{A_1} + S_{A_2} \geq S_A,$$

with the equality if and only if $\rho_A = \rho_{A_1} \otimes \rho_{A_2}$.

(iii) Strong subadditivity: for two areas $A$ and $B$ intersecting each other, we have

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}.$$

If two areas do not intersect with each other (i.e., $A \cap B = \emptyset$), this inequality is reduced to (ii).

In the past few years, it has been clarified that some important properties of a quantum ground state are encoded in the size-dependence of $S_\Omega$. For a system with short-range correlations only, $\Omega$ and $\bar{\Omega}$ correlate only in the vicinity of the boundary separating them and thus the entanglement entropy scales with the size of the boundary (boundary law) [70]. However, at a critical point with algebraically decaying correlations, the scaling of entanglement entropy exhibits a universal logarithmic correction characterizing the criticality. Specifically, in a one-dimensional quantum critical system described by a conformal field theory (CFT), the entanglement entropy on a subsystem of a length $L$ embedded in the infinite chain shows a logarithmic scaling law with a coefficient determined by the central charge $c$ of the CFT [73]:

$$S_{\Omega} \approx \frac{c}{3} \ln L + \text{const.}.$$  

In some two-dimensional quantum critical states, the entanglement entropy also contains a universal contribution, related to the geometry of the subsystem [15].

Another type of non-trivial entanglement was proposed by KP [31] and LW [41] for systems with topological order. It was argued that, if $\Omega$ is a disk (in a two-dimensional system) with a smooth boundary of length $L$, the entanglement entropy scales as

$$S_{\Omega} = \alpha L - \gamma + \cdots,$$

where the ellipsis represents terms which are negligible in the limit $L \to \infty$. If the area $\Omega$ is not a disk but has $m$ disconnected boundaries, the topological term $-\gamma$ in Eqn. (5.5) is multiplied by $m$. While the coefficient $\alpha$ depends on the microscopic details of the system, $\gamma$ is a universal constant characterizing topological order and was dubbed the topological entanglement entropy. Indeed, $\gamma$ measures the so-called total quantum dimension $D$ of topological order by $\gamma = \ln D$.

In the case of topological order described by a discrete gauge theory (e.g., $\mathbb{Z}_n$), $D$ is equal to the number of elements in the gauge group. In general, it is difficult to separate the topological term $-\gamma$ from the boundary term in Eqn. (5.5) because, on a lattice, the discrete nature of the boundary makes it difficult to define unambiguously the length $L$. However, KP and LW formulated some ways to define $\gamma$ by forming a linear combination of the entanglement entropies on plural areas sharing some boundaries, and cancelling the boundary terms out to leave the topological term. This proposal would be useful in detecting and classifying topological order from the ground-state wave functions. Conceptually, it demonstrates that topological order is a property of the ground-state wave function, not the Hamiltonian. Other properties of topological order, e.g., topological degeneracy and quasiparticle fractionalization, should be regarded as consequences of the entanglement structure in the ground-state wave function.

5.2 Numerical analysis for the triangular QDM

Here we numerically analyze the properties of entanglement entropy in the QDM on the triangular lattice, whose Hamiltonian is given by Eqn. (2.84). The phase diagram of this model is shown in Fig. 2.9 (b). The idea of KP and LW should apply to the dimer liquid phase of this model in $0.82(3) \lesssim v/t \lesssim 1$. The topological entropy for this phase is expected to be $\gamma = \ln 2 \simeq 0.6931$,
reflecting $\mathbb{Z}_2$ topological order in this phase \[54\]. We mainly consider the RK point \[66\], where the ground states are exactly known and where the calculation of reduced density matrices (and thus entanglement entropies) amounts to count the number of dimer coverings of the lattice satisfying some particular constraints. For $v/t < 1$, we perform Lanczos diagonalization of the Hamiltonian (2.84) for small systems (up to $N = 36$), and calculate the entanglement entropy for the obtained ground state. In particular, we examine the two constructions of topological entropy proposed by KP and LW. Furthermore, we consider the entanglement entropy on a particular topologically non-trivial area and design another procedure to measure $\gamma$, which, for the QDM, turns out to give an accurate value even in small systems.

We comment on related systems here. Kitaev’s model \[30\] is known to be the simplest solvable model with $\mathbb{Z}_2$ topological order, and the entanglement entropy for this model has been analyzed rigorously in Refs. \[19, 41\] and the value $\gamma = \ln 2$ for the topological entropy was confirmed. The solvable QDM (kagome lattice) of Ref. \[51\] can be mapped onto Kitaev’s model on the honeycomb lattice, and thus its entanglement entropy can be analyzed in the same way. These models give elegant results, but are too ideal for discussing generic features of topological order because they have a strictly zero spin-spin (or dimer-dimer in the QDM) correlation length and are completely free of finite-size effects. In this sense, our analysis on the QDM on the triangular lattice is a step toward more realistic systems: though we mainly consider the exact RK ground states, they have a finite dimer-dimer correlation length and finite-size effects arise. In a similar spirit but for another kind of topological order, the topological entanglement entropy of Laughlin wave functions was analyzed numerically in Ref. \[20\].

Computational settings are similar to those in Sec. 4.2.1; Lattices and RDMs are defined in the same way as there. Since the liquid phase under consideration exhibits degenerate ground states, we must specify for which state in the ground-state manifold we calculate the entanglement entropy. However, as long as the area is local, it was numerically demonstrated in Chap. 4 that the RDMs are identical for all states in the ground-state manifold, up to a correction which decays exponentially with the system size. Thus in this case we can take any state in the ground-state manifold. At the RK point $v/t = 1$, which we mainly consider in the following, we simply take the “equal-amplitude” state (2.83). The RDM of the “equal-amplitude” state can be calculated in a way described in the Appendix B, either by direct enumeration, or using Pfaffians. For $v/t < 1$, we use the ground state obtained by Lanczos diagonalization, which lies in the sector $p = ++$ for $N = 16$ and $p = --$ for $N = 36$.

### 5.2.1 Circular areas

We first consider the entanglement entropy on disks (areas with no holes) and discuss how the entanglement entropy scales with the extension of the area. Calculations were done for the RK wave function (2.83). As the choice of the area $\Omega$, we define circular areas in the following way: we draw a circle with a radius $R$ centred at a site or at an interior of a triangle and regard every bond whose midpoint is in the circle as an element of the area; see Fig. 4.5. This definition causes unavoidable ambiguity in the radius $R$; different radii can result in the same area. For example, the possible radius for the smallest site-centered area (consisting of six bonds) ranges in $0.5 < R < \sqrt{3}/2$. Here we analyze the data taking this ambiguity into account.

In Fig. 5.1, the values of $S_\Omega$ on circular areas are plotted versus the radius $R$. For each area, the data points from different system sizes (from $N = 16$ to 52) are placed at the minimum radius, and a horizontal bar is put for $N = 52$ to show the possible range of the radius. The data from different system sizes almost coincide, showing the smallness of the finite-size effects. We fit the data for $N = 52$ by a linear relation both for minimum and maximum radii. We observe

\[ D_\Omega \approx c e^{a R - bN}. \]

On the other hand, in the present case, the scaling (5.5) is quite sensitive to such ambiguity, as will be demonstrated.

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Footnote: In Sec. 4.2.2, we took the minimum radius in such a range. The ambiguity in radii is much smaller than the radii themselves and does not affect the obtained scaling $D_\Omega \approx c e^{a R - bN}$. On the other hand, in the present case, the scaling (5.5) is quite sensitive to such ambiguity, as will be demonstrated.
rough agreement with the linear fitting in both cases as expected from the proposed scaling form (5.5). The lines intersect with the vertical axis around $-0.1$ and $-1.8$ for the “minimum” and “maximum” cases respectively. These values sandwich the expected value $-\ln 2 \simeq -0.6931$ but are both away from it. We also tried fitting the data separately for site-centered and triangle-centered cases (not shown in the figure), but no essential difference was observed. These results seemingly contradict the proposed scaling form (5.5), but do not necessarily mean the failure of the proposal.

In general, on a lattice, the boundary of $\Omega$ is made of segments. If the sum of the segments is long enough, they contribute to the entanglement entropy by an amount proportional to the length. But in addition, we have to take into account the contribution coming from local correlations (between the regions $\Omega$ and $\tilde{\Omega}$) taking place in the vicinity of the angles between successive segments. If $\Omega$ is large, the contribution from these angles may be small (of order $O(L^0)$, compared to the boundary length $L$), but this contribution will still be of the same order as the topological term we are looking for. This is indeed demonstrated in the present calculation, where the ambiguity in the boundary length was taken into account as the ambiguity in the radius $R$. In fact, this is just one aspect of the ambiguity; considering the detailed shape of the boundary, the definition of the boundary length is more ambiguous. To compute $\gamma$ in a well-defined way, we need to turn to the construction using plural areas, which we discuss in the next subsection.

### 5.2.2 Construction of the topological entropy using plural areas

KP and LW proposed two ways to extract the topological constant $\gamma$ independent of the definition of the boundary length [31, 41]. The idea is to evaluate $\gamma$ by forming an appropriate linear combination of the entanglement entropies on different areas, so that the boundary contributions cancel out.
Kitaev-Preskill construction

In the KP construction [31], we consider a circle and divide it into three “fans”, $A$, $B$, and $C$. Then we form a linear combination

$$ S_{\text{topo}}^{\text{KP}} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}, \quad (5.6) $$

where $S_{XY...}$ denotes the entanglement entropy on a composite area $X \cup Y \cup \cdots$. In this combination, all the boundary contributions cancel out and a topological term $-\gamma$ is left. For example, let us consider the line separating $A$ and $B$. The boundary contributions along this line appears in $S_A$ and $S_B$ with a plus sign and in $-S_{BC}$ and $-S_{CA}$ with a minus sign, and they cancel out. Some attention needs to be paid to the triple point, around which four areas ($A$, $B$, $BC$, and $CA$) have different shapes and seemingly show different contributions. However, recalling Eqn. (5.2), the entropy on $BC$ is equal to that on the complement of $BC$, which has the same shape with $A$ in the vicinity of the triple point. Thus the contributions from $A$ and $BC$ in the vicinity of the triple point exactly match. The same argument applies to every line and every corner of boundaries, showing the exact cancellation of all the boundary contributions in Eqn. (5.6). Assuming the scaling (5.5), we expect $S_{\text{topo}}^{\text{KP}} = -\gamma$.

We apply this idea to our present model. We divide a circle by three lines emanating from the center as in Fig. 5.2. These lines are placed at angles $\theta_0 - 0$, $\theta_0 + 120^\circ - 0$ and $\theta_0 + 240^\circ - 0$ measured from the (reference) $u$ direction. Here “$-0$” represents an infinitesimal
Figure 5.4: Kitaev-Preskill topological entropy (5.6) as a function of $v/t$ for $N = 36$. In the large-$R$ limit, $S_{\text{KP topo}}^\text{} \overset{\text{topo}}{=}$ is expected to show a jump from $\ln 2$ in $\mathbb{Z}_2$ liquid phase $0.82(3) \lesssim v/t \leq 1$, to some positive value in the VBC phase $v/t \lesssim 0.82(3)$.

shift for avoiding collisions between the points and the boundaries. For example, points at an angle $\theta_0$ belong to $A$, not to $C$. We take $\theta_0 = 0^\circ$ or $30^\circ$ for site-centered circles (referred to as “S00” and “S30”) and $\theta_0 = 30^\circ$ or $90^\circ$ for triangle-centered circles (“T30” and “T90”). In these settings, the parts $A, B, C$ are equivalent under $120^\circ$ rotation, and we thus only need to calculate $S_{\text{KP topo}}^\text{} \overset{\text{topo}}{=} 3S_A - 3S_{AB} + S_{ABC}$.

We first consider the case of the RK wave function (2.83). In Fig. 5.3, the data of $S_{\text{KP topo}}^\text{}$ are plotted versus the radii $R$ of the circles. As in the case of circular areas presented in Fig. 5.1, finite-size effects are very small: except for the case where the circle $ABC$ occupies a substantial part of the system, the data from different $N$’s almost coincide. In the largest system $N = 52$, we can regard the data up to $R < 2.8$ as a good approximation of the values in the infinite system. In all the cases, $S_{\text{KP topo}}^\text{} \overset{\text{topo}}{=}$ decreases almost monotonically with $R$ and for large radii (specifically, $2.2 < R < 2.8$) shows values which are very close to $-\ln 2$, the expected value for a $\mathbb{Z}_2$ topologically ordered state.

Next we consider the region $v/t < 1$ of the Hamiltonian (2.84). In $\mathbb{Z}_2$ liquid phase $0.82(3) \lesssim v/t \leq 1$, $S_{\text{KP topo}}^\text{} \overset{\text{topo}}{=}$ is expected to show $-\ln 2$ in the large-$R$ limit. On the other hand, in $\sqrt{12} \times \sqrt{12}$ VBC phase $v/t \lesssim 0.82(3)$, where discrete symmetries are spontaneously broken, the finite-size ground state can be approximated by a linear superposition of $12$-fold symmetry-broken states. In such a state, we conjecture that the entanglement entropy on a disk $\Omega$ scales as $S_{\Omega} \simeq \alpha L + \ln d$ in the large-area limit, where $d$ is the ground-state degeneracy and is equal to 12 in the present case. The constant term $\ln d$ is not topological in the sense that the same value is obtained if $\Omega$ had another topology, unlike $-\gamma$ in Eqn. (5.5). Notice that this constant is positive, contrary to the topological term $-\gamma$. Assuming this, the combination (5.6) should give $\ln d$ in
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Figure 5.5: Division of annular areas for the Levin-Wen construction. Site-centered, \( R_{\text{in}} = 1.32, R_{\text{out}} = 3.12 \)

Figure 5.6: Topological entanglement entropy from the Levin-Wen construction at RK point.

a symmetry-broken phase. Thus, \( S_{\text{topo}}^{\text{KP}} \) is expected to jump from a negative (topological) value \(-\ln 2\) to a positive (non-topological) value along with the transition from the liquid phase to the VBC phase. We performed Lanczos diagonalization of the Hamiltonian (2.84) for a lattice with \( N = 36 \) (which is compatible with \( \sqrt{12} \times \sqrt{12} \) VBC ordering), and calculated \( S_{\text{topo}}^{\text{KP}} \) for the ground state, which lies in the sector \( p = - - \) in both the VBC and liquid phases on this lattice. The results are shown for every type of areas in Fig. 5.4. Because the system and area sizes are rather small, we do not observe a jump at the transition. However, we can still observe some tendency: for fixed \( v/t \), \( S_{\text{topo}}^{\text{KP}} \) tends to decrease as a function of \( R \) in the liquid side while it tends to increase in the VBC side. Some positive values of \( S_{\text{topo}}^{\text{KP}} \) in the VBC phase are also seen in “T30” case.

**Levin-Wen construction**

In the LW construction [41], we consider an annulus divided into four pieces as in Fig. 5.5, and form a combination

\[
S_{\text{topo}}^{\text{LW}} = S_{ABCD} - S_{ABC} - S_{CDA} + S_{AC}.
\] (5.7)

This combination is guaranteed to be non-positive from the strong subadditivity inequality (5.4) of entanglement entropies, namely,

\[
S_{\text{topo}}^{\text{LW}} = S_{X \cup Y} - S_X - S_Y + S_{X \cap Y} \leq 0,
\] (5.8)

where \( X = A \cup B \cup C \) and \( Y = C \cup D \cup A \). The combination (5.7) is expected to give \(-2\gamma\) for a topological phase and zero for a conventional phase (disordered, or with some symmetry-breaking order).

In Fig. 5.5, an annulus is divided by four lines at angles \( \theta_0 - 0, \theta_0 + 60^\circ + 0, \theta_0 + 180^\circ - 0, \theta_0 + 240^\circ + 0 \). We consider only site-centered annuli, and we set \( \theta_0 = 0^\circ \) or \( 30^\circ \) (again referred
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Figure 5.7: Zigzag area on a lattice with $T_1 = l_x u$ and $T_2 = l_y v$.

to as “S00” and “S30”). The result for the RK wave function is shown in Fig. 5.6. $R_{\text{in}}$ and $R_{\text{out}}$ denote the inner and outer radii of the annulus respectively, and $S_{\text{topo}}^{\text{LW}}$’s are plotted as a function of $R_{\text{out}}$. Up to $R_{\text{out}} < 3.2$, where the data for $N = 64$ well approximate the values in the infinite system, we observe that $S_{\text{topo}}^{\text{LW}}$ monotonically decreases with $R_{\text{out}}$ and approaches $-2 \ln 2$. Unfortunately, the convergence to $-2 \ln 2$ is not very clear up to this radius. In the LW construction, the requirement for the convergence is $\xi << R_{\text{in}}, R_{\text{out}} - R_{\text{in}}, L - 2R_{\text{out}}$, where $\xi$ is the correlation length ($\simeq 1$ at RK point) and $L = \sqrt{N}$ is the linear system size (or equivalently, the maximum possible $2R_{\text{out}}$). Thus, the LW construction suffers from stronger finite-area (not finite-$N$) effects than the KP construction which just requires $\xi << R, L - 2R$.

5.2.3 Zigzag area

We construct a different way to evaluate $\gamma$ using a thin “zigzag” area $\Omega$ winding around the torus as in Fig. 5.7. This area is invariant by translation in the $x$ direction and all points (black circles in Fig. 5.7) are equivalent by symmetry. In contrast to the more complicated areas considered before, we expect the boundary (i.e., non-topological) contribution to $S_\Omega$ to be precisely proportional to $l_x$, when $l_x$ is sufficiently larger than the correlation length $\xi$. In this new geometry, the thermodynamic behavior is obtained as soon as $\xi << l_x, l_y$, whereas the KP construction requires $\xi << R, L - 2R$, which is difficult to reach in exact diagonalization up to $N = 36$. Since the area is topologically non-trivial (it contains the incontractible cut $\Delta_1$), the value of $S_\Omega$ depends on the choice of the ground state, even for large systems. We calculate the entanglement entropies of this area in the ground-states $|\text{RK}\rangle$ and $|\text{RK}; p\rangle$ on isotropic lattices $l_x = l_y$, and write them as $S[\text{RK}]$ and $S[\text{RK}; p]$ respectively. The results are plotted in Fig. 5.8. As anticipated, $S_\Omega$ appears to be almost perfectly linear in $l_x$ (compared with the results of Fig. 5.1). Moreover, we observe that the topological constant $\gamma$ can be extracted in two different ways: a) by extrapolating (through a linear fit) $S[\text{RK}; p]$ at “$l_x = 0$” or b) by $-\gamma \simeq S[\text{RK}; p] - S[\text{RK}]$. These two follow from the scaling forms

$$S = \alpha_1 l_x \text{ for } |\text{RK}\rangle,$$

$$S = \alpha_1 l_x - \gamma \text{ for } |\text{RK}; p\rangle,$$

(5.9)

where $\alpha_1$ is a non-universal constant. A similar scaling was obtained rigorously by Hamma et al. [19] for a “ladder” area in Kitaev’s model on the square lattice. Here we confirmed that it holds accurately even in a system with a finite correlation length. The scaling forms (5.9) provide an accurate way to calculate the topological constant $\gamma$ even in relatively small systems. The condition (satisfied by QDM) is that topological sectors must be well defined and not mixed by the Hamiltonian, so that one can label the ground states by their sectors.
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Figure 5.8: Entanglement entropies on a zigzag area at the RK point. The upper line is a linear fit to $S_{[RK]}$. The lower one is a fit to $S_{[RK;p]}$ when $l_x = l_y$ is multiple of 4 and $S_{[RK;p]}$ otherwise. The topological constant estimated from the latter fit is $\gamma = 0.726 \pm 0.011$. This particular choice of $p$ as a function of $l_x$ is motivated by the fact that, when $v/t < 1$, it corresponds to the ground-state sector.

Figure 5.9: The topological term $-\gamma$ estimated using zigzag areas as a function of $v/t$. 
As another application, we utilize this to improve the evaluation of the topological term $-\gamma$ in the region $v/t < 1$. We performed Lanczos diagonalization for lattices with $l_x = l_y = 4$ and $l_x = l_y = 6$. For $v/t < 1$, the ground state lies in the sector $p = ++$ for $l_x = 4$ and $p = --$ for $l_x = 6$. We therefore compute the entropies $S[p = ++; l_x = 4]$ and $S[p = --; l_x = 6]$ on the zigzag areas and approximate the topological term $-\gamma$ by a linear extrapolation to $l_x = 0$. In the thermodynamic limit, the constant term extracted in this way is expected to jump from $-\ln 2$ to a positive value, as in the case of Fig. 5.4. However, $\sqrt{12} \times \sqrt{12}$ VBC ordering is compatible only with lattices where $l_x = l_y$ is a multiple of 6, and a linear relation $S_\Omega \simeq \alpha_1 l_x + \ln d$ holds only for such lattices. The lattice with $l_x = l_y = 4$ is out of this scaling, and thus the present estimation of the constant term is invalid for the VBC phase. We can still utilize the present estimation to confirm the stability of the topological term in the liquid phase, which is indeed seen in Fig. 5.9 better than in Fig. 5.4.

5.3 Conclusions

The concept of topological entanglement entropy was recently introduced by KP and LW as a way to detect and characterize topological order from a ground-state wave-function. We have illustrated numerically how this approach works in the case of the $\mathbb{Z}_2$ liquid phase of the QDM on the triangular lattice. We found that, due to lattice discretization, the topological entropy $\gamma$ cannot be obtained from a direct fit to the scaling form $S \simeq \alpha L - \gamma$. Instead, it is necessary to combine the entropies of several areas to cancel out the perimeter contributions, as suggested by KP and LW. In particular, for the KP construction, we clearly observed that in the large-area limit the topological entanglement entropy converges to the value expected for $\mathbb{Z}_2$ topological order. We also proposed a procedure to evaluate the topological entropy using a topologically non-trivial “zigzag” area, which gives an accurate value even in small systems. In addition to illustrating the concept of topological entanglement entropy in a “realistic” model, the present analysis offers an evidence of $\mathbb{Z}_2$ topological order in the QDM on the triangular lattice from a new perspective. Although the existence of topological degeneracy [53], and the analogy between this model and an Ising gauge theory [54] were already known, the present work confirms the $\mathbb{Z}_2$ structure in the ground-state wave function itself.

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2The zigzag area under consideration is not a disk (the width is too small to contain one unit cell of the $\sqrt{12} \times \sqrt{12}$ crystal) but all the $d = 12$ VBC patterns can be distinguished by some appropriate observable defined on this area. We thus expect the same scaling as disks.